Evaluation of an integral involving Airy functions



1985 J. Phys. A: Math. Gen. 18 2857

(http://iopscience.iop.org/0305-4470/18/14/034)

J. Phys. A: Math. Gen. **18** (1985) 2857-2858. Printed in Great Britain

**COMMENT**

# Evaluation of an integral involving Airy functions

L T Wille and J Vennik

Laboratorium voor Kristallografie en Studie van de Vaste Stof, Rijksuniversiteit Gent, Krijgslaan 281/S1, B-9000 Gent, Belgium

Received 16 April 1985

**Abstract.** We evaluate an integral which has occurred in recent studies of the asymptotic behaviour of the gap in the Mathieu equation.

The asymptotic behaviour of the regions of instability in the Mathieu equation has been studied recently by Harrell (1981, quoted by Avron and Simon (1981)). Both approaches yield the same answer, provided that the value of the integral

$$\mathscr{I} = \int_0^\infty \frac{\mathrm{Ai}(x)\mathrm{Bi}(x)}{(\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x))^2}\,\mathrm{d}x \tag{1}$$

where Ai and Bi are the standard Airy functions, is equal to $\pi/8$.

We give here a particularly straightforward derivation of this identity. Extensive use will be made of the standard tables of Abramowitz and Stegun (1965). To facilitate reference to a formula in this work it will be quoted as AS$n$ where $n$ refers to the appropriate formula number.

We start by rewriting the integral as

$$\mathscr{I} = \int_{-\infty}^0 \frac{\mathrm{Ai}(-x)\mathrm{Bi}(-x)}{(\mathrm{Ai}^2(-x)+\mathrm{Bi}^2(-x))^2}\,\mathrm{d}x. \tag{2}$$

Next, introducing the modulus and phase of the Airy functions (AS10.4.69),

$$\mathrm{Ai}(-x) = M(x)\cos\theta(x) \tag{3a}$$

$$\mathrm{Bi}(-x) = M(x)\sin\theta(x) \tag{3b}$$

we obtain

$$\mathscr{I} = \frac{1}{2}\int_{-\infty}^0 \frac{\sin 2\theta(x)}{M^2(x)}\,\mathrm{d}x. \tag{4}$$

This can be simplified using the following differential relation between modulus and phase (AS10.4.71),

$$M^2\theta' = -1/\pi. \tag{5}$$

Performing the integration, we finally obtain

$$\mathscr{I} = \tfrac{1}{4}\pi(\cos 2\theta(0) - \cos 2\theta(-\infty)). \tag{6}$$

0305-4470/85/142857+02\$02.25  

The phase of the Airy functions in the origin is (AS10.4.4)

$$\theta(0)=\pi/3 \tag{7}$$

and from the asymptotic behaviour (AS10.4.59, 64) we deduce

$$\theta(-\infty)=\pi/2. \tag{8}$$

Substitution of (7) and (8) into (6) gives

$$\mathcal{I}=\pi/8 \tag{9}$$

which is consistent with the results of Harrell (1981) and Avron and Simon (1981).

Obviously, in a completely similar way one can calculate the following indefinite integral,

$$\int \frac{\mathrm{Ai}(x)\mathrm{Bi}(x)}{(\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x))^2}\,\mathrm{d}x=\frac{\pi}{2}\frac{\mathrm{Ai}^2(x)}{\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x)} \tag{10}$$

where we have omitted the integration constant.

## References

Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
Avron J and Simon B 1981 *Ann. Phys., NY* **134** 76–84